An Efficient Finite-Element Formulation Without Spurious Modes for Anisotropic Waveguides

Istvan Bardi and Oszkar Biro

Abstract —**A numerically efficient finite-element formulation is presented for the analysis of lossless, inhomogeneously loaded, anisotropic waveguides of arbitrary shape. The electromagnetic field is described either by the three components of a magnetic vector potential and an electric scalar potential or by the three components of an electric vector potential and a magnetic scalar potential. The uniqueness of the potentials is ensured by the incorporation of the Coulomb gauge and by proper boundary conditions. Owing to the implementation of the solenoidality condition for the vector potential even in the case of zero wavenumber, no spurious modes appear. Variational expressions suited to the finite-element method are formulated in terms of the potentials. Standard finite-element techniques are employed for the numerical solution, leading to a generalized eigenvalue problem with symmetric, sparse matrices. This is solved by means of the bisection method with the sparsity of the matrices fully utilized. Dielectric- and ferrite-loaded waveguides with closed and open boundaries and including both isotropic and anisotropic materials are presented as examples.**

I. Introduction

SEVERAL finite-element formulations are in current use for solving lossless, inhomogeneously loaded, arbitrarily shaped waveguide problems including anisotropic material characteristics. The problem of the occurrence of nonphysical, spurious solutions can be considered to have been overcome. There are many ways of avoiding these spurious solutions. The efficiency of the finite-element codes depends to a large extent on the formulation used for getting rid of spurious modes.

The basic cause of the spurious modes lies in the inaccurate approximation of the zero eigenvalues and the corresponding eigenfunctions. The eigenfunctions belonging to the zero eigenvalue represent gradient fields satisfying the wave equation at zero wavenumber. Since the multiplicity of the zero eigenvalue is infinite, any given finite discretization, however fine, is bound to be incapable of approximating some strongly varying gradient eigenfunctions. The resulting inaccurate approximate eigenvalues are nonzero and are encountered as spurious modes that are difficult to distinguish from the truly nonzero wavenumbers. The divergence of the eigenfunctions belonging to the zero eigenvalue is not zero.

The key to the elimination of spurious solutions is the enforcement of the zero divergence of the vector field used for the description. The formulations differ in the way of incorporating this solenoidality condition.

Hayata *et al.* [3] employ the three components of the magnetic field intensity for the description. The z component of the magnetic field intensity is expressed by the transversal components from the condition of zero divergence. Thus, the Coulomb gauge is imposed on $\boldsymbol{H}$, and no spurious modes appear. The method has some disadvantageous properties: it is not applicable to the case of 3-D cavities, and the realization is rather complicated, leading to dense matrices with matrix inversions having to be performed.

Webb [7] imposes the Coulomb gauge by means of a kind of penalty factor method. There are no spurious modes and sparse matrix techniques are applicable but, as a consequence of the Lagrange multipliers, a time-consuming iteration method has to be employed for the solution of the generalized eigenvalue problem.

Hano [6] uses the three components of either the electric or the magnetic field intensity. The curl–curl operator is used. Special triangular elements ensure the continuity of the tangential components of the field vectors only with no constraint on the normal components. No spurious modes appear and there are as many zero eigenvalues as the number of unknowns representing the z component of the field. The implementation of the special triangular elements in a finite element code is rather complicated.

Svedin [9] uses all six components of the electric and magnetic field intensities. The first-order Maxwell equations are discretized and special measures are taken to enforce the interface conditions on the normal components. The divergence of the vectors applied is fixed implicitly to zero, so no spurious modes appear. The six variables and the additional interface conditions are the disadvantageous properties of the method. The finite-ele-

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ment code presented is valid for the case of lossy media, too.

A recently introduced possibility of avoiding the spurious solutions is the application of the edge element approximation [14]. The edge element approximation sets the multiplicity of the zero eigenvalue in the discretized problem to a finite number known in advance, so the zero eigenvalues can be approximated with good accuracy.

In this paper, a standard finite-element method based on nodal elements is presented. The three components of a vector potential together with a scalar potential are used for the description of the electromagnetic field. Two types of potentials are introduced: a magnetic vector potential with an electric scalar potential and an electric vector potential with a magnetic scalar potential. The Coulomb gauge is imposed based on the work of Biro *et al.* [1]. Owing to the uniqueness of the vector potential employed and to the fact that the solenoidality condition is ensured even in the case of zero wavenumber, no spurious modes appear. The method uses four scalar variables, a disadvantage offset by the simplicity of the finite element realization that leads to sparse matrices. Both the magnetic and the electric vector potential descriptions can be applied to both dielectric- and ferrite-loaded waveguides, so mixed structures can be analyzed by both methods. The method is valid for the case of anisotropic and lossy materials as well. Presently, an eight-node, isoparametric finite-element code is installed for the case of ideal, anisotropic materials. The examples presented serve to test the method and to show its applicability to various types of problems.

II. Potential Descriptions

A waveguide inhomogeneously loaded with anisotropic dielectric and magnetic materials is considered. The cross section, Ω, of arbitrary shape is in the $x-y$ plane and its boundary, Γ, consist partly of a perfect electric conductor and partly of a perfect magnetic wall. The relevant Maxwell equations for the time harmonic case are

$$\nabla \boldsymbol{H} = j\omega\epsilon_0[\epsilon_r]\boldsymbol{E} \tag{1}$$

$$\nabla \times \boldsymbol{E} = -j\omega\mu_0[\mu_r]\boldsymbol{H} \tag{2}$$

where $[\epsilon_r]$ and $[\mu_r]$ are the tensors of the relative permittivity and permeability, respectively.

For the description by a magnetic vector potential, $\boldsymbol{A}$, and an electric scalar potential, V ($\boldsymbol{A}, V$ formulation), the potentials are introduced as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{3}$$

$$\boldsymbol{E} = -j\omega\boldsymbol{A} - \nabla V'. \tag{4}$$

The following differential equation is to be satisfied by these potentials:

$$\nabla \times [\nu_r]\nabla \times \boldsymbol{A} - k_0^2[\epsilon_r]\boldsymbol{A} - k_0^2[\epsilon_r]\nabla V = \boldsymbol{0} \tag{5}$$

where

$$V' = j\omega V \tag{6}$$

$$[\nu_r] = [\mu_r]^{-1}. \tag{7}$$

The vector potential, $\boldsymbol{A}$, is not defined in a unique way by (3). Following [1], an additional term is added to (5) in order to fix the divergence of the vector potential to zero:

$$\nabla \times [\nu_r]\nabla \times \boldsymbol{A} - \nabla\nu_r\nabla\cdot\boldsymbol{A} - k_0^2[\epsilon_r]\boldsymbol{A} - k_0^2[\epsilon_r]\nabla V = \boldsymbol{0} \tag{8}$$

where

$$\nu_r = \frac{1}{3}\mathrm{Tr}[\nu_r]. \tag{9}$$

The constant ν_r is arbitrary, the choice (9) is based on dimensional reasons. "Tr" denotes the trace of the tensor. Since the equation

$$\nabla\cdot(j\omega\boldsymbol{D}) = 0 \tag{10}$$

no more follows from (8), this has to be prescribed explicitly:

$$\nabla\cdot\left(-k_0^2[\epsilon_r](\boldsymbol{A} + \nabla V)\right) = 0. \tag{11}$$

A further equation follows by taking the divergence of (8), provided (11) holds:

$$\nabla^2(\nu_r\nabla\cdot\boldsymbol{A}) = 0. \tag{12}$$

This consequence equation states that the Laplacian of $\nu_r\nabla\cdot\boldsymbol{A}$ vanishes. In order to make the vector potential obtained as the solution of the differential equations (8) and (11) unique, the following boundary conditions have to be fulfilled [1]:

$$\left.\begin{aligned} &\boldsymbol{A}\times\boldsymbol{n} = \boldsymbol{0} && (13)\\ &V = 0 && (14)\\ &\nu_r\nabla\cdot\boldsymbol{A} = 0 && (15)\end{aligned}\right\}\quad \text{on electric walls}$$

and

$$\left.\begin{aligned} &[\nu_r]\nabla\times\boldsymbol{A}\times\boldsymbol{n} = \boldsymbol{0} && (16)\\ &-k_0^2[\epsilon_r](\boldsymbol{A}+\nabla V)\cdot\boldsymbol{n} = 0 && (17)\\ &\boldsymbol{A}\cdot\boldsymbol{n} = 0. && (18)\end{aligned}\right\}\quad \text{on magnetic walls}$$

Note that the boundary conditions (13) and (14) state that the tangential component of the electric field intensity vanishes on electric walls, and conditions (16) and (17) set the tangential component of the magnetic field intensity and the normal component of the electric flux density to zero on magnetic walls. It can be shown [1] that these two condition imply that the normal derivative of $\nu_r\nabla\cdot\boldsymbol{A}$ is zero on magnetic walls, too. Hence, as a consequence of (12) and of boundary condition (15), the Coulomb gauge is satisfied by the vector potential $\boldsymbol{A}$. Its uniqueness then follows from the boundary conditions (13) and (18). The uniqueness condition of the vector potential coincides with the solenoidality condition, which is ensured even in the case of $k_0^2 = 0$. This means that when the vector potential represents a gradient field, not only its curl but also its divergence is zero; thus, in view of the boundary conditions (13) and (18), the vector potential itself vanishes, too. In this case, the ambiguity is represented by the scalar potential, V, being arbitrary. Obviously, any approximation of a scalar function by nodal elements

TABLE I
SUMMARY OF THE DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS OF DIFFERENT DESCRIPTIONS

Description	Differential Equations	Boundary Conditions: Electric Wall	Boundary Conditions: Magnetic Wall
A,V	$\nabla\times[\nu_r]\nabla\times A-\nabla\nu_r\nabla\cdot A-k_0^2[\epsilon_r]A-k_0^2[\epsilon_r]\nabla V=\mathbf{0}$	$A\times n=\mathbf{0}$	$[\nu_r]\nabla\times A\times n=\mathbf{0}$
		$V=0$	$[\epsilon_r](A+\nabla V)\cdot n=0$
		$\nu_r\nabla\cdot A=0$	$A\cdot n=0$
	$\nabla\cdot(-k_0^2[\epsilon_r](A+\nabla V))=0$		
F,ψ	$\nabla\times[\kappa_r]\nabla\times F-\nabla\kappa_r\nabla\cdot F-k_0^2[\mu_r]F-k_0^2[\mu_r]\nabla\psi=\mathbf{0}$	$[\kappa_r]\nabla\times F\times n=\mathbf{0}$	$F\times n=\mathbf{0}$
		$[\mu_r](F+\nabla\psi)\cdot n=0$	$\psi=0$
		$F\cdot n=0$	$\kappa_r\nabla\cdot F=0$
	$\nabla\cdot(-k_0^2[\mu_r](F+\nabla\psi))=0$		

TABLE II
SUMMARY OF THE DIFFERENT FUNCTIONALS AND BOUNDARY CONDITIONS

Description	Functional	Electric Wall: Natural	Electric Wall: Dirichlet	Magnetic Wall: Natural	Magnetic Wall: Dirichlet
A,V	$I(A,V)=\int_\Omega\nabla\times A^*[\nu_r]\nabla\times A+\nabla\cdot A^*\nu_r\nabla\cdot A-k_0^2(A^*[\epsilon_r]A+A^*[\epsilon_r]\nabla V+\nabla V^*[\epsilon_r]A+\nabla V^*[\epsilon_r]\nabla V)\,d\Omega$	$\nu_r\nabla\cdot A=0$	$A\times n=\mathbf{0}$, $V=0$	$[\nu_r]\nabla\times A\times n=\mathbf{0}$, $[\epsilon_r](A+\nabla V)\cdot n=0$	$A\cdot n=0$
F,ψ	$I(F,\psi)=\int_\Omega\nabla\times F^*[\kappa_r]\nabla\times F+\nabla\cdot F^*\kappa_r\nabla\cdot F-k_0^2(F^*[\mu_r]F+F^*[\mu_r]\nabla\psi+\nabla\psi^*[\mu_r]F+\nabla\psi^*[\mu_r]\nabla\psi)\,d\Omega$	$[\kappa_r]\nabla\times F\times n=\mathbf{0}$, $[\mu_r](F+\nabla\psi)\cdot n=0$	$F\cdot n=0$	$\kappa_r\nabla\cdot F=0$	$F\times n=\mathbf{0}$, $\psi=0$

describes an exact gradient field, so there is no reason for spurious solutions arising from the discretization.

Indeed, if the differential equations (8) and (11) are solved with the boundary conditions (13)–(15) on electric walls and (16)–(18) on magnetic walls, no spurious modes are present in the finite-element solution.

For the description in terms of an electric vector potential, F, and a magnetic scalar potential, ψ (F,ψ formulation), the potentials are introduced as

$$D=\nabla\times F \tag{19}$$

$$H=j\omega F-\nabla\psi'. \tag{20}$$

The same discussion can be carried out as in the case of A,V formalism.

The differential equations and boundary conditions in the different formulations are summarized in Table I.

III. FUNCTIONAL FORMULATION AND FINITE ELEMENT REALIZATION

The differential equations and boundary conditions are of the same form for both the A,V and the F,ψ description. Therefore, only the notations of the A,V formulation will be used, but the results are directly applicable to the F,ψ case as well.

For the solution of differential equations (8) and (11), the following functional has to be extremized:

$$I(A,V)=\int_\Omega(\nabla\times A^*[\nu_r]\nabla\times A+\nabla\cdot A^*\nu_r\nabla\cdot A)\,d\Omega-k_0^2\int_\Omega(A+\nabla V)[\epsilon_r](A+\nabla V)\,d\Omega. \tag{21}$$

The first variation of $I(A,V)$ has to be zero with respect to both A^* and V^*:

$$\delta_{A^*}I(A,V)=\int_\Omega\big[\nabla\times[\nu_r]\nabla\times A-\nabla\nu_r\nabla\cdot A-k_0^2[\epsilon_r](A+\nabla V)\big]\cdot\boldsymbol{\delta}A^*\,d\Omega-\oint_\Gamma([\nu_r]\nabla\times A\times n)\cdot\boldsymbol{\delta}A^*\,d\Gamma+\oint_\Gamma\nu_r\nabla\cdot An\cdot\boldsymbol{\delta}A^*\,d\Gamma=0 \tag{22}$$

$$\delta_{V^*}I(A,V)=\int_\Omega k_0^2[-\nabla\cdot[\epsilon_r](A+\nabla V)]\delta V^*\,d\Omega+\oint_\Gamma k_0^2[\epsilon_r](A+\nabla V)\cdot n\delta V^*\,d\Gamma=0. \tag{23}$$

Conditions (22) and (23) of zero variation are fulfilled, provided that the differential equations (8) and (11) to be solved are satisfied and boundary condition (15) on electric walls and boundary conditions (16) and (17) on magnetic walls hold. These are the natural boundary conditions of the functional (21). Evidently, the remaining Dirichlet boundary conditions, (13), (14), and (18), must be satisfied explicitly.

A summary of the different functionals and boundary conditions is found in Table II.

In the following, advantage is taken of the fact that every field quantity varies in a special way with the coordinate z:

$$Q(x,y,z)=Q(x,y)e^{-j\beta z} \tag{24}$$

where β is the propagation coefficient. In this case, the vector and scalar potentials can be written as

$$A(x,y,z)=\left[A_x(x,y)\mathbf{e}_x+A_y(x,y)\mathbf{e}_y + jA_z(x,y)\mathbf{e}_z\right]e^{-j\beta z} \tag{25}$$

$$V(x,y,z)=V(x,y)e^{-j\beta z}. \tag{26}$$

Standard two-dimensional finite-element techniques can be used for the discretization. In the present implementation, eight-node isoparametric finite elements have been employed. The components of the vector potential, as well as those of the scalar potential, are continuous by the nature of the finite-element method. The continuity of the appropriate field components follows from the continuity of the potentials and from the natural interface conditions of the functional (21). Performing the discretization, the following generalized eigenvalue problem is obtained:

$$\left\{\begin{bmatrix} \mathbf{M}_{AA} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} - k_0^2\begin{bmatrix} \mathbf{N}_{AA} & \mathbf{N}_{AV} \\ \mathbf{N}_{VA} & \mathbf{N}_{VV}\end{bmatrix}\right\}\begin{bmatrix} \mathbf{A} \\ \mathbf{V}\end{bmatrix}=\mathbf{0}. \tag{27}$$

The matrices $\mathbf{M}$ and $\mathbf{N}$ are symmetric and sparse. The matrix $\mathbf{M}$ is, however, singular; as seen in (27) it has as many zero rows and columns as the number of scalar potential unknowns (n_V). Thus, the first nonzero wavenumber is the (n_V+1)th eigenvalue of (27). The zero eigenvalues correspond to the eigenvectors describing vanishing vector potentials and arbitrary scalar potentials and are obtained as exactly zero during the numerical solution.

The eigenvalue problem has been solved by the bisection method [13], with the sparsity of the matrices fully utilized.

IV. Examples

The first example is an embedded dielectric waveguide with anisotropic material characteristics. The region was bounded by electric walls at a distance of about $5w$. It was assumed that there was no propagation in the transversal plane at the values of β investigated. The number of degrees of freedom in the finite-element calculation was 772. The dispersion characteristics of the first mode are shown in Fig. 1(b), and the agreement with the curve of Koshiba [4] is satisfactory. The distribution of the propagating power is illustrated in Fig. 1(c) by the lines of the constant z component of the Poynting vector for the same mode.

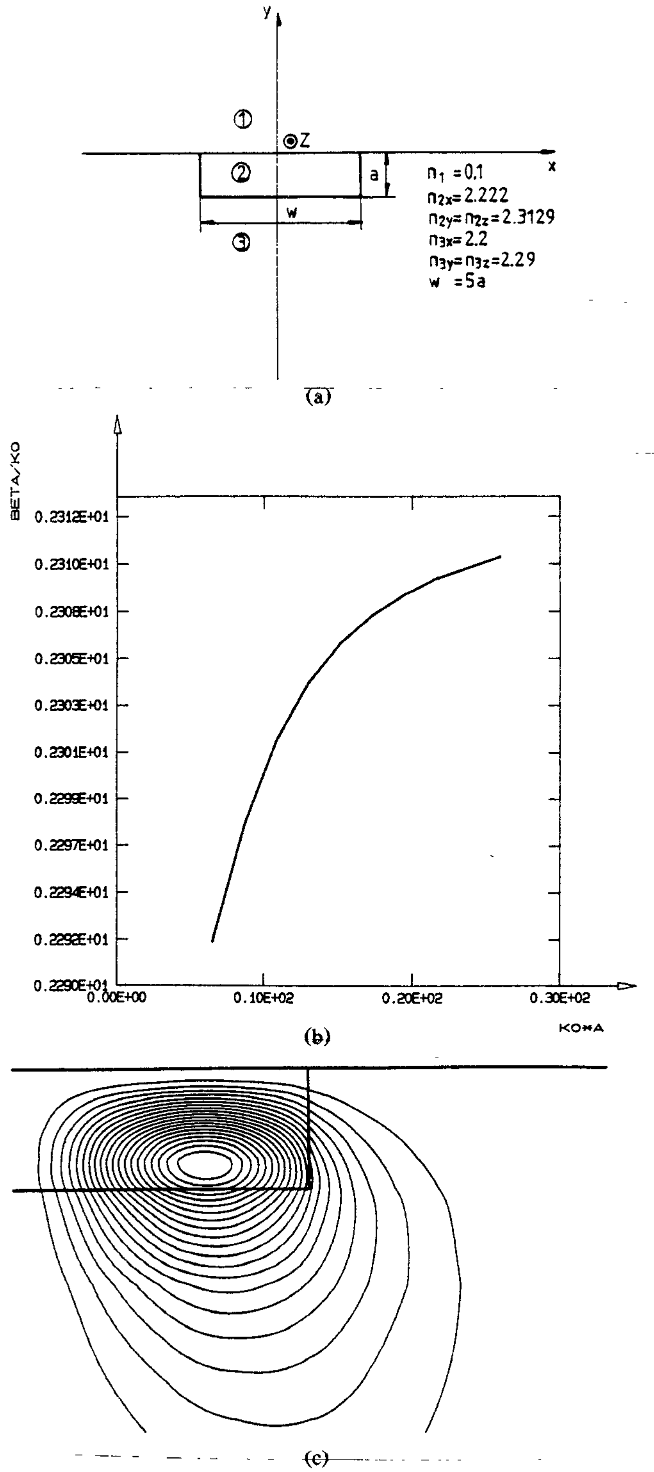

Fig. 1. (a) Embedded dielectric waveguide. (b) Dispersion characteristics of an embedded dielectric waveguide. (c) Distribution of power propagating in the z direction in an embedded dielectric waveguide.

The second example is a ferrite-loaded waveguide of rectangular cross section. The relative permittivity and permeability have been chosen based on the work of Hano [6]. The values are

$$\epsilon_{2r}=10 \qquad [\mu_{2r}]=\begin{bmatrix} 0.875 & 0.0 & -j0.375 \\ 0.0 & 1.0 & 0.0 \\ j0.375 & 0.0 & 0.875\end{bmatrix}.$$

TABLE III
COMPARISON OF WAVENUMBERS OBTAINED BY THE TWO FORMULATIONS FOR A FERRITE-LOADED RECTANGULAR WAVEGUIDE

Formulation:	A,V		F,ψ	
Degrees of Freedom:	424		536	
βa	$k_{01}a$	$k_{02}a$	$k_{01}a$	$k_{02}a$
−1	0.8925	1.483	0.8925	1.485
0	0.8095	1.438	0.8095	1.440
1	0.9700	1.492	0.9700	1.494

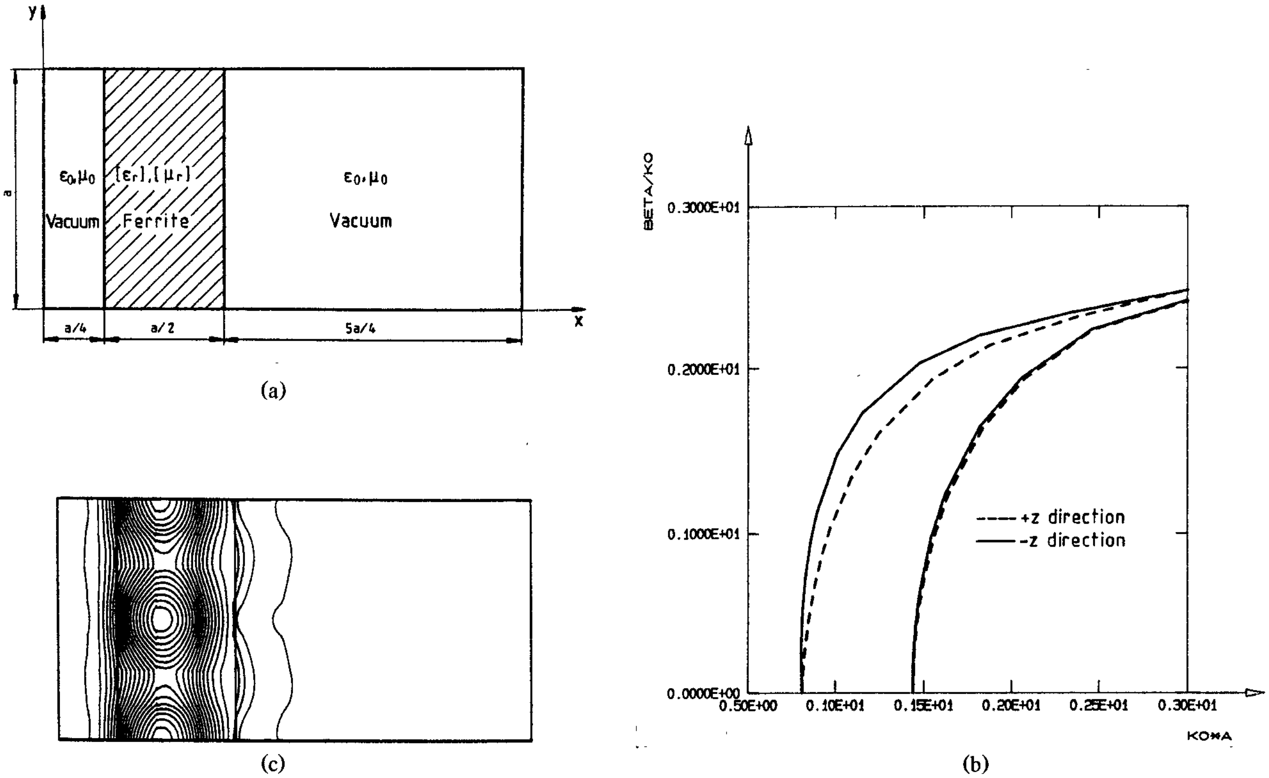

Fig. 2. (a) Ferrite-loaded waveguide. (b) Dispersion characteristics of an anisotropic ferrite-loaded waveguide. (c) Distribution of power propagating in the z direction in an anisotropic ferrite-loaded waveguide. Third mode, $\beta/k_0 = 1.90$, $k_0 a = 2.91$.

In general, the permeability tensor depends on the frequency. In our example (based on [6]), the frequency dependence was neglected, it was assumed that the permeability tensor is approximately constant in the frequency range studied. The description presented is not applicable for the frequency-dependent case because k_0^2 is treated as an eigenvalue.

Information on the discretization and a comparison of the A,V and F,ψ version are given in Table III. The dispersion curves of the first two modes for both positive and negative propagation have been computed and compared with the results of Hano [6]. The curves are shown in Fig. 2(b); the agreement is good. The lines of the constant z component of the Poynting vector are plotted in Fig. 2(c) for the third mode.

The third example is an inhomogeneously loaded waveguide of elliptic cross section. The material characteristics are described by the following tensors:

$$[\epsilon_{r1}] = \langle 10.0 \quad 10.0 \quad 10.0 \rangle$$

$$[\mu_{r1}] = \begin{bmatrix} 0.875 & 0.0 & -j0.375 \\ 0.0 & 1.0 & 0.0 \\ j0.375 & 0.0 & 0.875 \end{bmatrix}$$

$$[\epsilon_{r2}] = \begin{bmatrix} 2.25 & 0.0 & 0.0 \\ 0.0 & 2.25 & 0.0 \\ 0.0 & 0.0 & 1.5 \end{bmatrix} [\mu_{r2}] = \langle 1 \quad 1 \quad 1 \rangle.$$

The discretization and a comparison between the two versions are given in Table IV. The dispersion characteristics of the first two modes are plotted in Fig. 3(b). The lines of the constant z component of the Poynting vector are shown in Fig. 3(c), for the third mode.

TABLE IV
COMPARISON OF WAVENUMBERS OBTAINED BY THE TWO FORMULATIONS FOR A DIELECTRIC- AND FERRITE-LOADED WAVEGUIDE OF ELLIPTIC CROSS SECTION

Formulation:	$\boldsymbol{A}, V$		$\boldsymbol{F}, \psi$	
Degrees of Freedom:	548		612	
βa	$k_{01}a$	$k_{02}a$	$k_{01}a$	$k_{02}a$
−6	2.3585	2.3959	2.3603	2.3962
0	0.8057	1.2753	0.8051	1.2757
6	2.2234	2.5376	2.2242	2.5388

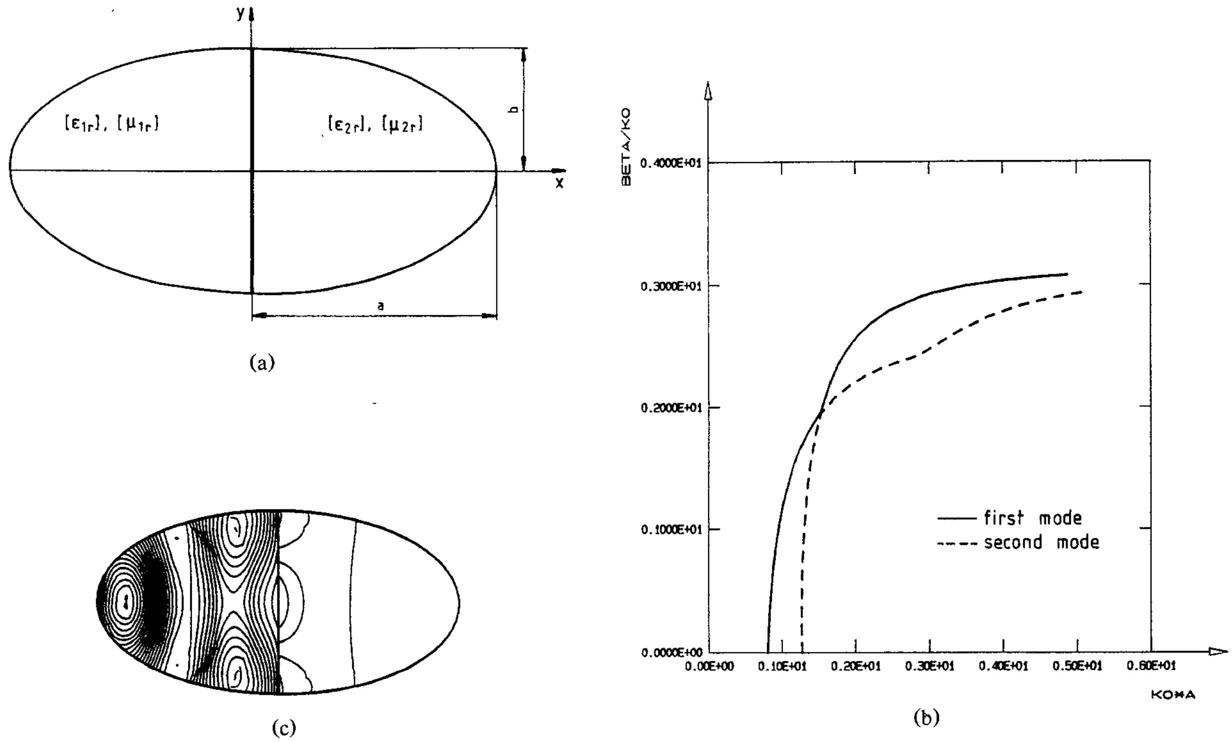

Fig. 3. (a) Dielectric and ferrite loaded waveguide of elliptic cross section. (b) Dispersion characteristics of a dielectric and ferrite loaded waveguide of elliptical cross section. (c) Distribution of power propagating in the z direction in a dielectric and ferrite loaded waveguide of elliptical cross section. Third mode, $\beta / k_0 = 1.41$, $k_0 a = 70.800$.

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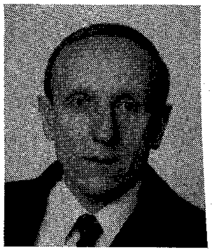

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